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Abstract
Leonard pairs (LP) are matrices with the property of mutual tri-diagonality. We introduce and study a classical analogue of LP. We show that corresponding classical 'Leonard' dynamical variables satisfy non-linear relations of the AW-type with respect to Poisson brackets.

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1. Introduction

Let $F, G, \ldots$ be classical dynamical variables (DV) that can be represented as differentiable functions of the canonical finite-dimensional variables $q_i, p_i, i = 1, 2, \ldots, N$.

The Poisson brackets (PB) $\{F, G\}$ are defined as [1]

$$\{F, G\} = \sum_{i=1}^{N} \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}. \tag{1.1}$$

In particular, for canonical variables one has standard PB [1]

$$\{q_i, p_j\} = \delta_{ij} \qquad \{q_i, q_j\} = \{p_i, p_j\} = 0.$$

The PB satisfies fundamental properties [1].

(i) PB is a linear function in both F and G;
(ii) PB is anti-symmetric $\{F, G\} = -\{G, F\}$;
(iii) PB satisfies the Leibnitz rule $\{F_1F_2, G\} = F_1\{F_2, G\} + F_2\{F_1, G\}$;
(iv) For any dynamical variables F, G, H PB satisfies the Jacobi identity $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$.

Properties (i)–(iv) are trivial consequences of definition (1.1). It is possible, however, to construct abstract PB starting from the axioms (i)–(iv).

PB are important in classical mechanics because they determine time dynamics: if the DV H is a Hamiltonian of the system, then for any DV G one has the Poisson equation

$$\dot{G} = \{G, H\}. \tag{1.2}$$

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In particular, the DV F is called an *integral* if it has zero PB with the Hamiltonian $\{F, H\} = 0$. In this case F does not depend on t.

In many problems of classical mechanics DV form elegant algebraic structures which are closed with respect to PB. For example, let $H = \mathbf{p}^2/2 + U(r)$ be the Hamiltonian describing motion of a particle in a central field with the potential $U(r)$ depending only on a distance $r = \left(q_1^2 + q_2^2 + q_3^2\right)^{1/2}$. In this case the components of the angular momentum $\boldsymbol{M} = [\mathbf{r}, \mathbf{p}]$ are integrals: $\{M_i, H\} = 0, i = 1, 2, 3$. The variables M_i themselves form a classical Poisson $so(3)$ algebra:

$$\{M_i, M_k\} = \epsilon_{ikj} M_j \tag{1.3}$$

where ϵ_{ikj} is a standard completely antisymmetric tensor and summation is assumed with respect to repeated subscript j.

Algebra (1.3) is the simplest example of the so-called linear Poisson structures (or Poisson Lie algebras): in all such structures the PB of several basic DV are expressed as linear functions of these DV.

A less trivial example of self-closed Poisson structures provides the Kepler problem with the potential $U(r) = -\alpha/r$. In this case (in addition to the standard angular momentum $\boldsymbol{M}$) there exists another integral of motion—the so-called Laplace vector $\boldsymbol{A} = [\boldsymbol{M}, \mathbf{p}] + \alpha\mathbf{q}/r$. It is easily verified that the components of the two vectors $\boldsymbol{M}$ and $\boldsymbol{A}$ are closed in frames of the Poisson algebra [14]

$$\{M_i, M_j\} = \epsilon_{ijk} M_k \qquad \{M_i, A_j\} = \epsilon_{ijk} A_k \qquad \{A_i, A_j\} = -2H\epsilon_{ijk} M_k \tag{1.4}$$

where H is the Hamiltonian of the Kepler problem. In this case the PB of the components A_i, A_j is not a linear function of basic integrals $\boldsymbol{M}, \boldsymbol{A}$. However, one can linearize this algebra if we fix the value of the energy $E = H$. This leads again to linear Poisson Lie algebras $so(4)$, $so(3, 1)$ and $e(3)$ depending on the value of the energy E (for details see [14]).

The Poisson structures with *non-linear* PB were discussed in [15, 11]. Sklyanin introduced [15] the so-called quadratic Poisson algebra consisting of four DV S_0, S_1, S_2, S_3 such that PB $\{S_i, S_k\}$ is expressed as a quadratic function of the generators S_i. The Sklyanin algebra appears quite naturally from the theory of algebraic structures related to the Yang–Baxter equation in mathematical physics. Sklyanin also proposed to study general non-linear Poisson structures. Assume that there exist N dynamical variables $F_i, i = 1, 2, \dots, N$, such that PB of these variables are closed in frames of the non-linear relations

$$\{F_i, F_k\} = \Phi_{ik}(F_1, \dots, F_N) \qquad i, k = 1, 2, \dots, N \tag{1.5}$$

where $\Phi_{ik}(F_1, \dots, F_N)$ are (non-linear, in general) functions of N variables.

Several interesting examples of such non-linear Poisson structures are described in [11].

In [7] another example of such non-linear Poisson algebra was proposed. This example is connected with the property of 'mutual integrability' and leads to the so-called classical AW relations, where the abbreviation AW means 'Askey–Wilson algebra'. Indeed, as was shown in [17], the operator (i.e. non-commutative) version of AW relations has a natural representation in terms of generic Askey–Wilson polynomials, introduced in [2] (see also [12]).

The motivation of the present work was the concept of the 'Leonard pairs' proposed in [10, 16]. Two $N \times N$ matrices X, Y form a Leonard pair if there exist invertible matrices S and T such that the matrix $S^{-1}XS$ is diagonal whereas the matrix $S^{-1}YS$ is irreducible tri-diagonal and similarly, the matrix $T^{-1}YT$ is diagonal whereas the matrix $T^{-1}XT$ is irreducible tri-diagonal. We will call such a property 'mutual tri-diagonality'. Leonard showed [13] that the eigenvalue problem for a Leonard pair X, Y leads to the q-Racah polynomials (for a definition see, e.g., [12]).

Terwilliger showed [10, 16] that a Leonard pair X, Y satisfies a certain algebraic relation with respect to commutators. In turn, the Terwilliger relations follow from the so-called relations of the AW algebra studied in [17, 7].

In this paper we study a classical analogue of the 'Leonard pair' property. We introduce classical dynamical variables X and Y satisfying this property and show that X and Y should satisfy classical Poisson AW relations [7].

2. Algebraic relations for the classical 'Leonard pair'

In this section, we will assume that all dynamical variables are functions of the two independent canonical variables q, p with the main relation $\{q, p\} = 1$. In what follows we will assume that q, p are complex numbers and all corresponding complex-valued functions $X(q, p)$, $Y(q, p), \ldots$ are assumed to be analytic (in the usual sense of theory of functions of several complex variables [6]) in some open domain D of the complex space C^2.

We say that DV $X(q, p)$ and $Y(q, p)$ are *independent* if

$$\{X, Y\} \equiv \frac{\partial(X, Y)}{\partial(q, p)} \neq 0 \tag{2.1}$$

in some open domain D, where $\frac{\partial(X,Y)}{\partial(q,p)}$ is the Jacobian. Of course, the initial variables q, p are independent. Functions $X(q, p)$, $Y(q, p)$ define some map from the open domain D in the complex space (q, p) to the open domain $\tilde{D}$ in the space of complex variables X, Y. As is well known [6] condition (2.1) means that in the domain $\tilde{D}$ it is possible to introduce inverse map $q = q(X, Y)$, $p = p(X, Y)$ which is also analytic and maps $\tilde{D}$ to D.

Recall that by canonical transformation (CT) it is assumed an analytical transformation $(q, p) \to (x, y)$ to a pair of new canonical variables x, y. This means that $x = x(q, p)$, $y = y(q, p)$ are analytic functions in variables q, p and moreover

$$\{x, y\} = \frac{\partial(x, y)}{\partial(q, p)} = 1. \tag{2.2}$$

Clearly, the new variables x, y are independent.

Definition. *We say that two independent DV $X(q, p)$ and $Y(q, p)$ form a classical Leonard pair (CLP) if there exist two canonical transformations (defined in corresponding open domains of C^2) $(q, p) \to (x, y)$ and $(q, p) \to (\xi, \eta)$ (with $\{x, y\} = \{\xi, \eta\} = 1$) such that in variables x, y the DV X, Y take the form*

$$X = \phi(x) \qquad Y = A_1(x)\exp(y) + A_2(x)\exp(-y) + A_3(x) \tag{2.3}$$

while in the variables ξ, η we have

$$Y = \psi(\xi) \qquad X = B_1(\xi)\exp(\eta) + B_2(\xi)\exp(-\eta) + B_3(\xi) \tag{2.4}$$

where $\phi(x)$, $A_i(x)$, $\psi(x)$, $B_i(x)$ are some analytical functions.

Let us explain the origin of our definitions (2.3) and (2.4). If one replaces X, Y with non-commuting operators $\hat{X}, \hat{Y}$ then the Leonard duality means that there exists a representation in which $\hat{X}$ is diagonal whereas $\hat{Y}$ is tri-diagonal. Assume that the operators $\hat{X}, \hat{Y}$ are in general infinite dimensional. Then the diagonal representation for $\hat{X}$ means that there exists a realization on a space of functions $f(x)$ of a variable x such that $\hat{X}$ acts as a multiplication operator $\hat{X}f(x) = \phi(x)f(x)$ with some prescribed function $\phi(x)$. Then on the same functional space the tri-diagonal operator $\hat{Y}$ acts as

$\hat{Y} f(x) = A_1(x) f(x+h) + A_2(x) f(x-h) + A_3(x) f(x)$ where h is a complex parameter. Symbolically, the operator $\hat{Y}$ can be presented in the form

$$\hat{Y} = A_1(x) \exp(\mathrm{i}h\hat{y}) + A_2(x) \exp(-\mathrm{i}h\hat{y}) + A_3(x) \tag{2.5}$$

where $\hat{y} = -\mathrm{i}\partial_x$ is the canonical momentum operator satisfying the standard commutation relation $[x, \hat{y}] = \mathrm{i}$.

Vice versa, the Leonard pair condition means that there exists a dual representation on the same functional space such that $\hat{S}^{-1}\hat{Y}\hat{S} f(x) = \psi(x) f(x)$ and

$$\hat{S}^{-1}\hat{X}\hat{S} = B_1(x) \exp(\mathrm{i}h\hat{y}) + B_2(x) \exp(-\mathrm{i}h\hat{y}) + B_3(x) \tag{2.6}$$

where $\hat{S}$ is an operator providing 'diagonalization' of the operator Y.

Then it is natural to define CLP by replacing operators $x, \hat{y}$ with their classical conjugate canonical variables x, y or ξ, η. In other words, we formally replace:

(i) all operators $\hat{X}, \hat{Y}, \ldots$ with *commuting* classical dynamical variables $X, Y, \ldots$.
(ii) commutators $[\hat{X}, \hat{Y}]$ with PB (X, Y) (this is the well-known Dirac procedure of the correspondence between quantum and classical mechanics [11]).

We thus arrive at our definitions (2.3) and (2.4).

Note that the concept of the Leonard pair is closely related to the so-called bispectrality problem [5]. We thus arrive also at the classical analogue of the bispectral problem.

Before analysing algebraic relations between X, Y supposed by the property of CLP, let us find conditions under which variables X, Y are independent. We first calculate Poisson bracket of the variables X, Y in the representation (2.3):

$$Z = \{X, Y\} = \phi'(x)(A_1(x)\exp(y) - A_2(x)\exp(-y)). \tag{2.7}$$

Independence of X, Y means that $Z \neq 0$ in some domain D of two complex variables x, y. Hence in order for variables X, Y to be independent it is necessary that $\phi'(x) \neq 0$ and at least one of the functions $A_1(x)$ and $A_2(x)$ is non-zero inside some open domain D_x in the complex plane x. Moreover, we assume that domain D lies apart from complex curve defined by $A_1(x)\exp(y) - A_2(x)\exp(-y) = 0$.

Analogously, we assume that $\psi'(\eta) \neq 0$ and at least one of the functions $B_1(\xi)$ and $B_2(\xi)$ is non-zero inside some domain D_ξ in the complex plane ξ.

As a byproduct we have that the function $\phi(x)$ is invertible in the domain D_x as well as the function $\psi(\xi)$ is invertible in the domain D_ξ.

Now we have obviously

$$Z^2 = (\phi'(x))^2 \left((Y - A_3(x))^2 - 4A_1(x)A_2(x)\right). \tag{2.8}$$

As the function $\phi(x)$ is invertible in D_x, one can express $x = \phi^{(-1)}(X)$, where $\phi^{(-1)}(x)$ is a function inverted with respect to $\phi(x)$.

Then (2.8) means that Z^2 is a quadratic function of Y:

$$Z^2 = V_1(X)Y^2 + V_2(X)Y + V_3(X) \tag{2.9}$$

with some yet unknown (analytic) functions $V_i(X)$.

Quite analogously we can calculate Z^2 using representation (2.4):

$$Z^2 = (\psi'(\xi))^2 \left((X - B_3(\xi))^2 - 4B_1(\xi)B_2(\xi)\right). \tag{2.10}$$

Again, as $\psi(\xi)$ is an invertible function, in D_ξ we have the relation

$$Z^2 = W_1(Y)X^2 + W_2(Y)X + W_3(Y) \tag{2.11}$$

i.e. Z^2 is a quadratic function of X with unknown functions $W_i(Y)$.

Note that relations (2.11), (2.9) are valid in some open domain $\tilde{D}$ of the space of two complex variables X, Y. Comparing (2.9) and (2.11) we arrive at the functional equation

$$V_1(X)Y^2 + V_2(X)Y + V_3(X) = W_1(Y)X^2 + W_2(Y)X + W_3(Y) \tag{2.12}$$

which should be valid for any values of two independent complex variables X, Y inside the domain $\tilde{D}$.

Lemma 1. *Functional equation (2.12) is valid only if the functions $V_i(x)$, $W_i(x)$, $i = 1, 2, 3$, are polynomials of degree not exceeding 2.*

Proof. We use the fact that X, Y are independent variables in the domain $\tilde{D}$. Choose three arbitrary non-coinciding values y_1, y_2, y_3 of Y belonging to this domain. Then (2.12) can be considered as a system of three different linear equations for three unknown functions $V_1(x), V_2(x), V_3(x)$, where x is an independent variable. The determinant of this system is non-zero (because y_1, y_2, y_3 are distinct) and right-hand sides are polynomials of degree at most 2. Hence this system has a unique solution yielding $V_i(x)$, $i = 1, 2, 3$, as polynomials in x of degree at most 2. Due to obvious symmetry between X and Y we can equally obtain that $W_i(x)$ are also polynomials in x of degree at most 2. □

We thus proved that Z^2 should be a non-zero polynomial in two variables X, Y having degree at most 2 with respect to each variable, i.e.

$$Z^2 = \alpha_1 X^2Y^2 + \alpha_2 X^2 Y + \alpha_3 XY^2 + \alpha_4 X^2 + \alpha_5 Y^2 + \alpha_6 XY + \alpha_7 X + \alpha_8 Y + \alpha_9 \tag{2.13}$$

with some parameters α_i, $i = 1, 2, \ldots, 9$.

From (2.13) we can conclude that the Poisson brackets $\{Z, X\}$ and $\{Y, Z\}$ are closed in frames of the classical AW-algebra. Indeed, we have

$$0 = \{\alpha_9, X\} = \{Z^2 - F(X, Y), X\} = Z(2\{Z, X\} + F_Y(X, Y)) \tag{2.14}$$

where

$$F(X, Y) = \alpha_1 X^2Y^2 + \alpha_2 X^2Y + \alpha_3 XY^2 + \alpha_4 X^2 + \alpha_5 Y^2 + \alpha_6 XY + \alpha_7 X + \alpha_8 Y \tag{2.15}$$

and $F_Y(X, Y)$ denotes partial derivative with respect to Y.

Note that $Z \neq 0$ as X, Y are independent. Hence we can conclude

$$\{X, Z\} = \tfrac{1}{2} F_Y(X, Y) = Y\left(\alpha_1 X^2 + \alpha_3 X + \alpha_5\right) + \left(\alpha_2 X^2 + \alpha_6 X + \alpha_8\right)/2. \tag{2.16}$$

Quite analogously we obtain

$$\{Z, Y\} = \tfrac{1}{2} F_X(X, Y) = X\left(\alpha_1 Y^2 + \alpha_2 Y + \alpha_4\right) + \left(\alpha_3 Y^2 + \alpha_6 Y + \alpha_7\right)/2. \tag{2.17}$$

Thus we get that three dynamical variables X, Y and $Z = \{X, Y\}$ form Poisson algebra with relations (2.16) and (2.17). It is easily verified that the Jacobi identity

$$\{X, \{Y, Z\}\} + \{Z, \{X, Y\}\} + \{Y, \{Z, X\}\} = 0$$

holds for this algebra. Introduce the dynamical variable

$$K = Z^2 - F(X, Y) \tag{2.18}$$

where $F(X, Y)$ is given by (2.15). It is clear from previous considerations that K plays the role of the *Casimir element* of our algebra, i.e.

$$\{K, X\} = \{K, Y\} = \{K, Z\} = 0.$$

The Poisson algebra with relations (2.16) and (2.17) was introduced in [7] and is called the classical AW algebra (i.e. Askey–Wilson algebra). As was noted in the introduction, generic Poisson algebras with non-linear Poisson brackets were considered in [15, 11]. In the 'quantum' case (i.e. when X and Y are operators) the AW relations were considered in [17, 7].

Remark. If $\alpha_1 = \alpha_2 = \alpha_3 = 0$ (i.e. the function $F(X, Y)$ has degree at most 2) then AW algebra is reduced to a Lie–Poisson algebra:

$$\begin{aligned}\{Z, X\} &= -\alpha_5 Y - (\alpha_6 X + \alpha_8)/2 \\ \{Y, Z\} &= -\alpha_4 X - (\alpha_6 Y + \alpha_7)/2\end{aligned} \tag{2.19}$$

which is linear with respect to generators X, Y. The constant terms α_8, α_7 in rhs of (2.19) can be avoided by shifts $X \to X + \text{const}$, $Y \to Y + \text{const}$.

We mention also a remarkable property of the classical AW algebra [7]. Assume that X is chosen as Hamiltonian: $H = X$. Then we have $\dot{Y} = \{Y, H\} = -Z$. From (2.13) we then obtain $\dot{Y}^2 = F(H, Y) + \alpha_9 =$ quadratic in Y. Hence $Y(t)$ is an *elementary function* in the time t. This means that $Y(t) = G_1(H)\exp(\omega(H)t) + G_2(H)\exp(-\omega(H)t) + G_3(H)$ or $Y(t) = G_1(H)t^2 + G_2(H)t + G_3(H)$, where $G_i(H), \omega(H)$ are some functions in the Hamiltonian H. Due to obvious symmetry between X, Y, the same property holds if one chooses Y as Hamiltonian: $H = Y$. In this case $X(t)$ behaves as elementary function in the time t. This property was called 'mutual integrability' in [7]. It can be considered as classical analogues of the property of 'mutual tri-diagonality' [16, 10] in the 'quantum' case.

We thus have

Proposition 1. *If dynamical variables X, Y form CLP then they should satisfy Poisson AW algebra (2.16), (2.17).*

Moreover we have

Proposition 2. *Let $X(q, p), Y(q, p)$ denote independent classical dynamical variables and put $Z = \{X, Y\}$. Then the following are equivalent:*

(i) X, Y, Z satisfy algebraic relations (2.17), (2.16);
(ii) $\{K, X\} = \{K, Y\} = 0$, where $K = Z^2 - F(X, Y)$, and where $F(X, Y)$ has expression (2.15).

Proof. We already showed (ii) → (i). The inverse statement is verified directly using the Leibnitz rule for Poisson brackets. □

Remark. If $\{K, X\} = \{K, Y\} = 0$ for some dynamical variable K then $\{K, Z\} = 0$ as follows from the Jacobi identity.

3. Representations of the AW relations

In the previous section we showed that AW relations are a *necessary* condition for two variables $X(q, p), Y(q, p)$ to form CLP. In this section we show that this condition is *sufficient* under some non-degeneracy restriction upon the parameters α_i.

Definition. *We say that* AW *relations (2.17), (2.16) are non-degenerate if at least one of the three parameters α_1, α_3, α_5 is non-zero, and at least one of the three parameters α_1, α_2, α_4 is non-zero. This can be written in a concise form as*

$$|\alpha_1|^2 + |\alpha_3|^2 + \left|\alpha_5^2\right| \neq 0 \qquad |\alpha_1|^2 + |\alpha_2|^2 + |\alpha_4|^2 \neq 0. \tag{3.1}$$

Proposition 3. *If two independent variables $X(q, p), Y(q, p)$ satisfy non-degenerate AW relations (2.17), (2.16), then the variables X, Y form CLP.*

Proof. By proposition 2 relations (2.17), (2.16) for two independent variables X, Y are equivalent to the existence of a non-zero polynomial $F(X, Y) = \alpha_1 X^2 Y^2 + \cdots + \alpha_8 Y$ in two variables X, Y such that $\{K, X\} = \{K, Y\} = 0$, where $K = Z^2 - F(X, Y)$.

So, assume that there exist nine parameters $\alpha_1, \ldots, \alpha_9$ such that

$$Z^2 = \alpha_1 X^2 Y^2 + \cdots + \alpha_8 Y + \alpha_9 \tag{3.2}$$

where non-degeneracy conditions (3.1) are fulfilled. Choose a new variable (x', y') (not necessarily canonical) such that $X = x'$. Then $Y = \Phi(x', y')$ with some function $\Phi(x', y')$. The function $\Phi(x', y')$ can be chosen in such a manner that the variable y' is canonical conjugate with respect to x', i.e. $\{x', y'\} = 1$. Indeed, from (3.2) it is sufficient that function $\Phi(x', y')$ should satisfy the differential equation:

$$\Phi^2_{y'}(x', y') = \pi_1(x')\Phi^2(x', y') + \pi_2(x')\Phi(x', y') + \pi_3(x') \tag{3.3}$$

where

$$\pi_1(x) = \alpha_1 x^2 + \alpha_3 x + \alpha_5 \qquad \pi_2(x) = \alpha_2 x^2 + \alpha_6 x + \alpha_8 \qquad \pi_3(x) = \alpha_4 x^2 + \alpha_7 x + \alpha_9. \tag{3.4}$$

Condition (3.1) means that $\pi_1(x) \neq 0$. Then from (3.3) it follows that

$$\Phi(x', y') = a_1(x')\exp(\sigma(x')y') + a_2(x')\exp(-\sigma(x')y') + a_0(x') \tag{3.5}$$

where

$$\sigma^2(x) = \pi_1(x) \tag{3.6}$$

and

$$a_0(x) = -\frac{\pi_2(x)}{2\pi_1(x)} \qquad 4a_1(x)a_2(x) = \frac{\pi_2^2(x)}{4\pi_1^2(x)} - \frac{\pi_3(x)}{\pi_1(x)}. \tag{3.7}$$

Note that the rational function $a_0(x)$ is determined uniquely, while the functions $a_1(x), a_2(x)$ are determined only through their product.

Now we can change canonical variables $(x', y') \to (x, y)$ in such a manner that

$$x' = \phi(x) \qquad y' = y/\phi'(x)$$

where $\phi(x)$ is some function and $\phi'(x)$ is its derivative. The fundamental relation $\{x, y\} = 1$ is fulfilled automatically. Choose the function $\phi(x)$ as a solution of the differential equation

$$\phi'^2(x) = \pi_1(\phi(x)) = \alpha_1\phi^2(x) + \alpha_3\phi(x) + \alpha_5 \tag{3.8}$$

(clearly, solution of this equation is elementary and depends on α_i, $i = 1, 3, 5$, see below). As $\pi_1(x) \neq 0$, the function $\phi(x)$ is not a constant. Taking into account $\sigma^2(x) = \pi_1(x)$ we have in new variables x, y

$$X = \phi(x) \qquad Y = A_1(x)\,e^y + A_2(x)\,e^{-y} + A_0(x) \tag{3.9}$$

where $A_i(x) = a_i(\phi(x)), i = 1, 2, 3$.

But then we return to condition (2.3).

Note that the function $\phi(x)$ can be easily found from (3.8)

(i) if $\alpha_1 \neq 0$ then $\phi(x) = C_1\,e^{\omega x} + C_2\,e^{-\omega x} + C_3$, where $\omega^2 = \alpha_1$, $C_3 = -\alpha_3/(2\alpha_1)$, $C_3^2 - 4C_1C_2 = \alpha_5/\alpha_1$;

(ii) if $\alpha_1 = 0$ and $\alpha_3 \neq 0$ then $\phi(x) = C_1x^2 + C_2x + C_3$, where $C_1 = \alpha_3/4$, $C_2^2 - 4C_1C_3 = \alpha_5$.

(iii) if $\alpha_1 = \alpha_3 = 0$ and $\alpha_5 \neq 0$ then $\phi(x) = \sqrt{\alpha_5}x + C$, where C is an arbitrary constant.

The function $A_0(x)$ is determined uniquely:

$$A_0(x) = -\frac{\pi_2(\phi(x))}{2\pi_1(\phi(x))} \tag{3.10}$$

while for $A_1(x)$ and $A_2(x)$ only their product $V(x)$ is determined uniquely

$$V(x) \equiv A_1(x)A_2(x) = \frac{D(\phi(x))}{16\pi_1^2(\phi(x))} \tag{3.11}$$

where

$$D(\phi) = \pi_2^2(\phi) - 4\pi_1(\phi)\pi_3(\phi) \tag{3.12}$$

is a polynomial in ϕ of degree at most 4.

If $D(\phi) = 0$ then $\Phi(x, y)$ is

$$\Phi(x, y) = A_1(x)\,\mathrm{e}^{y} + A_0(x) \qquad \text{or} \qquad \Phi(x, y) = A_2(x)\,\mathrm{e}^{-y} + A_0(x) \tag{3.13}$$

where $A_1(x)$ and $A_2(x)$ are arbitrary analytical functions. The reason that only $V(x)$ is determined uniquely can easily be explained. Indeed, there exists the obvious canonical transformation $x \to x, y \to y + f(x)$, where $f(x)$ is an arbitrary function. Such transformation preserves the 'tri-diagonality' of the variable Y (2.3) but changes the functions $A_{1,2}(x)$: $A_1 \to A_1\,\mathrm{e}^{f(x)}$ and $A_2 \to A_1\,\mathrm{e}^{-f(x)}$. Clearly, the product $V(x) = A_1(x)A_2(x)$ is invariant with respect to such canonical transformation.

We can use this freedom to reduce $\Phi(x, y)$ to a standard form. For example, we can always choose coordinates in such a way that

$$\Phi(x, y) = \mathrm{e}^{y} + V(x)\,\mathrm{e}^{-y} + A_0(x) \tag{3.14}$$

or

$$\Phi(x, y) = 2\sqrt{V(x)}\cosh(y) + A_0(x) \tag{3.15}$$

for $D(\phi) \neq 0$ and

$$\Phi(x, y) = \mathrm{e}^{y} + A_0(x) \tag{3.16}$$

for $D(\phi) = 0$.

Note that $\phi(x)$, $A_0(x)$, $V(x)$ are elementary functions of the argument x.

Using symmetry between X and Y in AW relations we see that there exists a canonical transformation $(q, p) \to (\xi, \eta)$ such that $Y = \psi(\xi)$, $X = B_1(\xi)\,\mathrm{e}^{\eta} + B_2(\xi)\,\mathrm{e}^{-\eta} + B_3(\eta)$ where $\psi(\xi)$ satisfies the equation

$$\psi'^2(x) = \alpha_1\psi^2(x) + \alpha_2\psi(x) + \alpha_4$$

and functions $B_i(\xi), i = 1, 2, 3$, have the expressions similar to (3.7). Hence starting from AW-relations (2.16), (2.17) we proved that there exist two canonical transformations $(q, p) \to (x, y)$ and $(q, p) \to (\xi, \eta)$ with the properties (2.3), (2.4).

We thus proved proposition 3. □

Remark. From our considerations it follows that canonical variables x, y can take all possible complex values (apart from points where $\phi'(x) = 0$ and $A_1(x) = A_2(x) = 0$ and points belonging to the curve $A_1(x)\exp(y) - A_2(x)\exp(-y) = 0$). Inverse functions $x = x(X, Y), y = y(X, Y)$ are defined uniquely on smaller domains but one can use the standard technique of Riemannian surfaces to define corresponding analytic functions. The same is valid for variables ξ, η.

From relations (2.17) and (2.16) we obtain as an obvious consequence the following relations

$$\{X, \{X, \{X, Y\}\}\} = \left\{X, \alpha_1X^2Y + \alpha_3XY + \alpha_5Y\right\} \tag{3.17}$$

and

$$\{Y, \{Y, \{Y, X\}\}\} = \left\{Y, \alpha_1 Y^2 X + \alpha_2 XY + \alpha_4 X\right\}. \tag{3.18}$$

Relations (3.17) and (3.18) are exact classical analogies of those obtained in [16] to describe Leonard pairs (in [16] relations PB {, } are replaced with commutators [,]).

A non-trivial question is: whether relations (3.17) and (3.18) are sufficient to determine CLP?

The answer is positive for one-dimensional representations of Poisson algebra generated by (3.17) and (3.18), i.e. when $X(q, p)$ and $Y(q, p)$ depend only on two canonical variables q, p.

Indeed, assume that two independent DV $X(q, p)$, $Y(q, p)$ satisfy relations (3.17), (3.18). Then it is possible to choose new canonical variables (x, y) such that $X = x,\ Y = \Phi(x, y)$. From (3.17) we then obtain the equation upon the function $\Phi(x, y)$

$$\Phi_{yyy}(x, y) = \left(\alpha_1 x^2 + \alpha_3 x + \alpha_5\right)\ \Phi_y(x, y)$$

($\Phi_y(x, y)$ denotes partial derivative with respect to y). This equation has general solution

$$\Phi(x, y) = C_1(x)\, \mathrm{e}^{\sigma(x)y} + C_2(x)\, \mathrm{e}^{-\sigma(x)y} + C_3(x) \tag{3.19}$$

where $C_1(x), C_2(x), C_3(x)$ are arbitrary functions of the argument x and $\sigma(x)$ is given by (3.6). Performing canonical transformations $x \to \phi(x),\ y \to y/\phi'(x)$ with $\phi(x)$ given by (3.8) we arrive at the representation (2.3). Due to symmetry between X, Y this means that for one-dimensional representations the variables X, Y satisfy relations (2.3) and (2.4) and hence form a classical Leonard pair.

We thus arrive at:

Proposition 4. *Assume that $X(q, p)$, $Y(q, p)$ are two independent DV and $Z = \{X, Y\}$. Then the following statements are equivalent:*

(i) *$X(q, p)$ and $Y(q, p)$ form a CLP;*
(ii) *$X(q, p)$, $Y(q, p)$ and $Z(q, p)$ satisfy AW relations (2.17), (2.16) with conditions (3.1);*
(iii) *$X(q, p)$ and $Y(q, p)$ satisfy relations (3.17), (3.18) with the same conditions.*

Note that if $\alpha_1 = \alpha_2 = \alpha_3 = 0$ then the Terwilliger relations (3.17), (3.18) become so-called classical Dolan–Grady relations [4, 3]:

$$\{X, \{X, \{X, Y\}\}\} = \alpha_5\{X, Y\} \qquad \{Y, \{Y, \{Y, X\}\}\} = \alpha_4\{Y, X\}. \tag{3.20}$$

It is seen from (2.17) and (2.16) that for one-dimensional representations the classical Dolan–Grady relations are reduced to some algebra with *linear* PB

$$\{X, Y\} = Z \qquad \{Z, X\} = -(\alpha_5 Y + \alpha_6 X/2 + \alpha_8/2) \qquad \{Y, Z\} = -(\alpha_4 X + \alpha_6 Y/2 + \alpha_7/2). \tag{3.21}$$

Shifting generators $X \to X + \beta_1$, $Y \to Y + \beta_2$ with appropriately chosen constants β_1, β_2 we can get rid of the constants α_7, α_8.

4. Degenerate Leonard pairs

So far, we assumed that the AW algebra was non-degenerate, i.e. at least one of the parameters $\alpha_1, \alpha_3, \alpha_5$ (and also $\alpha_1, \alpha_2, \alpha_4$) is non-zero.

In this section we consider the degenerate case. Assume that

$$\alpha_1 = \alpha_3 = \alpha_5 = 0 \tag{4.1}$$

and at least one of the parameters $\alpha_2, \alpha_6, \alpha_8$ is non-zero. This means that $\pi_1(x) \equiv 0$ but $\pi_2(x) \neq 0$.

We are seeking representations of the AW relations in new canonical variables x, y such that $X = \phi(x)$, $Y = \Phi(x, y)$ such that

$$\phi'^2(x) = \pi_2(\phi(x))/2 = \left(\alpha_2\phi^2(x) + \alpha_6\phi(x) + \alpha_8\right)/2. \tag{4.2}$$

Then for $\Phi(x, y)$ we have the equation

$$\Phi_y^2(x, y) = 2\Phi(x, y) + 2\frac{\pi_3(\phi)}{\pi_2(\phi)}. \tag{4.3}$$

The general solution of this equation is

$$\Phi(x, y) = Y = (y + f(x))^2/2 + u(x) \tag{4.4}$$

where $f(x)$ is an arbitrary function of x and

$$u(x) = -\frac{\pi_3(\phi(x))}{\pi_2(\phi(x))}. \tag{4.5}$$

It is possible to put $f(x) = 0$ by an appropriate trivial canonical transformation $x \to x$, $y \to y - f(x)$. Then we have

$$Y = y^2/2 + u(x). \tag{4.6}$$

That is, in the degenerate case Y can be reduced to the ordinary non-relativistic one-particle Hamiltonian $y^2/2 + u(x)$ with the potential $u(x)$.

If one has $\pi_1(x) \equiv \pi_2(x) \equiv 0$ but $\pi_3(x) \neq 0$ then it is possible to get a representation of the AW algebra in the form

$$X = \phi(x) \qquad Y = y + f(x) \tag{4.7}$$

where $\phi(x)$ satisfies the equation

$$\phi'^2(x) = \pi_3(\phi(x)) \tag{4.8}$$

and $f(x)$ is an arbitrary function. Again it is possible to put $f(x) = 0$ by the same canonical transformation. That is, in this case we have that Y coincides with the momentum y.

Note that in both cases the function $\phi(x)$ has the expression of the form $\phi(x) = C_1\,\mathrm{e}^{\omega x} + C_2\,\mathrm{e}^{-\omega x} + C_3$ or $\phi(x) = C_1x^2 + C_2x + C_3$ where at least one of the constants C_1, C_2 is non-zero.

Definition. *We will call a generalized classical Leonard pair (GCLP) a pair of independent classical dynamical variables $X(q, p), Y(q, p)$ such that there exist two canonical transformations $(q, p) \to (x, y)$ and $(q, p) \to (\xi, \eta)$ such that in coordinates (x, y) one has $X = \phi(x)$, and Y is either $A_1(x)\,\mathrm{e}^y + A_2(x)\,\mathrm{e}^{-y} + A_3(x)$ or $A_1(x)y^2 + A_2(x)y + A_3(x)$ with some functions $\phi(x), A_i(x), i = 1, 2, 3$ (in both cases it is assumed that at least one of the functions $A_1(x)$ or $A_2(x)$ is non-zero). Analogously, in coordinates (ξ, η) one has $Y = \psi(\xi)$ and X is either $B_1(\xi)\,\mathrm{e}^\eta + B_2(\xi)\,\mathrm{e}^{-\eta} + B_3(\xi)$ or $B_1(\xi)\eta^2 + B_2(\xi)\eta + B_3(\xi)$ with some functions $\psi(\xi), B_i(\xi), i = 1, 2, 3$ (again at least one of the functions $B_1(x)$ or $B_2(x)$ is non-zero).*

It is easy to show (repeating previous considerations) that if variables X, Y form GCLP then they satisfy AW relations.

We thus have

Proposition 5. *Two statements are equivalent:*

(i) variables $X(q, p), Y(q, p)$ form GCLP;
(ii) variables $X(q, p), Y(q, p)$ and $Z = \{X, Y\}$ satisfy AW relations (2.16), (2.17).

This proposition gives complete characterization of the GCLP.

We would like to comment on the relation between non-degenerate $Y = A_1(x)\,\mathrm{e}^y + A_2(x)\,\mathrm{e}^{-y} + A_3(x)$ and degenerate $Y = A_1(x)y^2 + A_2(x)y + A_3(x)$ realizations. By a canonical transformation $x \to \kappa^{-1}x,\ p \to \kappa p$ one can rewrite the non-degenerate realization in the form $Y = A_1(x;\kappa)\,\mathrm{e}^{\kappa p} + A_2(x;\kappa)\,\mathrm{e}^{-\kappa p} + A_3(x;\kappa)$. In the limit $\kappa \to 0$ we can, in principle, obtain degenerate realization (if the functions $A_i(x;\kappa)$ behave appropriately in this limit).

5. Multidimensional representations

So far, we have restricted ourselves to the one-dimensional representations of corresponding algebraic relations. It is natural to ask whether multi-dimensional representations can provide another (more complicated) algebraic structure, which does not coincide with AW relations.

The answer is positive and we show this using the simplest example of the Dolan–Grady algebra. The generic case of multi-dimensional representations should be considered separately.

It is well known that the Dolan–Grady relations [4]

$$[A_1,[A_1,[A_1,A_0]]] = 16[A_1,A_0] \qquad [A_0,[A_0,[A_0,A_1]]] = 16[A_0,A_1] \tag{5.1}$$

generate an infinite-dimensional linear Onsager algebra [3]:

$$\begin{aligned} &[A_n, A_m] = 4G_{n-m} \\ &[G_n, A_m] = 2A_{n+m} - 2A_{m-n} \\ &[G_n, G_m] = 0 \qquad n = 0,\ \pm 1, \pm 2, \ldots. \end{aligned} \tag{5.2}$$

In these relations [,] denotes commutator of the operators A_m, G_m. 'One-dimensional representations' of this algebra are equivalent to representations of sl_2 algebra; however, there are 'multi-dimensional' representations as well which can be presented as a direct sum of a finite number of irreducible representations of sl_2 algebra. These representations cannot be reduced to one-dimensional ones. For details see, e.g., [3].

We consider the classical analogue of this statement. By appropriate scaling of the variables X, Y we can choose the following classical analogue of the Dolan–Grady relations

$$\{X,\{X,\{X,Y\}\}\} = -\{X,Y\} \qquad \{Y,\{Y,\{Y,X\}\}\} = -\{Y,X\}. \tag{5.3}$$

Let $J_a^{(i)}, a = 1,2,3, i = 1,2,\ldots,N$, be a set of independent classical sl_2 Poisson algebras satisfying relations

$$\left\{J_a^{(i)}, J_b^{(k)}\right\} = \delta_{ik}\,\epsilon_{abc} J_c^{(i)}. \tag{5.4}$$

We choose the following realization of the relations (5.3)

$$X = \sum_{i=1}^{N} \alpha_i J_1^{(i)} + \beta_i J_2^{(i)} \qquad Y = \sum_{i=1}^{N} \gamma_i J_1^{(i)} + \delta_i J_2^{(i)}. \tag{5.5}$$

Then it is easily verified that relations (5.3) hold if

$$\alpha_i = \cos\theta_i \qquad \beta_i = \sin\theta_i \qquad \gamma_i = \cos\phi_i \qquad \delta_i = \sin\phi_i \tag{5.6}$$

where ϕ_i and θ_i, $i = 1,2,\ldots,N$, are arbitrary parameters. If $N = 1$ we return to the special case of the one-dimensional representation of the Dolan–Grady relations. In this case X, Y and Z themselves satisfy sl_2 relations as expected. However for $N > 1$ and arbitrary ϕ_i, θ_i it is impossible to reduce realization (5.5) to simple sl_2 relations. This means that classical Dolan–Grady relations admit multi-dimensional representations which cannot be reduced to one-dimensional ones. One can expect that a similar situation occurs for the general case of relations (3.17), (3.18). The classification of algebraic relations admitting non-trivial multi-dimensional representations is an interesting open problem.

6. Examples of classical Leonard pairs

In this section we give some explicit examples of GCLP.

Our first example is quite elementary and is based on the classical Lie–Poisson algebra sl_2 defined by the relations

$$\{N_0, N_\pm\} = \pm N_\pm \qquad \{N_-, N_+\} = 2N_0. \tag{6.1}$$

The Casimir element of this algebra is

$$N^2 = N_0^2 - N_+N_-. \tag{6.2}$$

We start from a realization of the algebra (6.1) by canonical variables q, p

$$N_- = q^2p + 2\nu q \qquad N_+ = p \qquad N_0 = qp + \nu \tag{6.3}$$

with an arbitrary parameter ν. In the realization (6.3) the Casimir element takes the value $N^2 = \nu^2$.

Let us take

$$X = N_0 - \nu = qp \qquad Y = N_- + N_+ = q^2p + 2\nu q + p. \tag{6.4}$$

In this case $Z = p - q^2p - 2\nu q$. It is seen that variables X, Y are independent (i.e. $Z \neq 0$) in a whole C^2 apart from a set of complex points (q, p) belonging to the complex curve $p(1 - q^2) = 2\nu q$. So it is sufficient to take any open domain D not belonging to this curve.

Hence

$$Z^2 = Y^2 - 4X^2 - 8\nu X.$$

This means that we have a non-degenerate Leonard pair.

It is easily verified that representation $X = x, Y = \mathrm{e}^y + (x^2 + 2\nu x)\,\mathrm{e}^{-y}$ provides a canonical transformation from the variables q, p to new canonical variables x, y. Similarly, the representation $X = \mathrm{e}^\eta + \frac{(\nu^2-\xi^2)}{4}\,\mathrm{e}^{-\eta} - \nu, \quad Y = 2\mathrm{i}\xi$ provides a canonical transformation from q, p to the canonical variables ξ, η.

It is interesting to note that X, Y are *linear* functions of the momentum p. On the other hand, Y is a combination of two exponents $\exp(\pm y)$ from the new momentum y.

Consider now examples of degenerate CLP. The first example is trivial: a pair of canonical variables $X = q, Y = p$ form degenerate CLP. The canonical variables x, y coincide with q, p and the dual variables are $\xi = -p, \eta = q$.

The second example of degenerate Leonard pair is the choice: $X = q, Y = p^2 + q^2$. Note that Y coincides with the Hamiltonian of the harmonic oscillator [1]. In this example the variables x, y merely coincide with q, p because Y is already a quadratic function of $p = y$. Calculating $Z^2 = 4p^2 = 4(Y - X^2)$ we see that X, Y are independent for all complex values q, p apart from $p = 0$. The variable Z satisfies condition (2.13) and hence X, Y indeed form degenerate CLP. Omitting elementary calculations, we give the dual picture in the coordinates ξ, η: $X = \exp(\eta) + \mathrm{i}\xi/2\exp(-\eta), Y = 2\mathrm{i}\xi$.

This example can be generalized in the following way.

Let us take

$$X = \phi(x) \qquad Y = y^2/2 + u(x) \tag{6.5}$$

with some functions $\phi(x)$ and $u(x)$. Such a choice corresponds to our definition of degenerate CLP. Note that Y in the form (6.5) coincides with the Hamiltonian of one-dimensional motion in the potential $u(x)$ [1]. Consider, when X, Y form a Leonard pair. We have $Z = \{X, Y\} = y\phi'(x)$. On the other hand, from (2.13) we should have

$$Z^2 = y^2\phi'^2(x) = \alpha_2X^2Y + \alpha_4X^2 + \alpha_6XY + \alpha_7X + \alpha_8Y + \alpha_9 \tag{6.6}$$

(clearly $\alpha_1 = \alpha_3 = \alpha_5 = 0$ because all terms in (2.13) containing these coefficients have degree more than 2 with respect to the variable y).

From (6.6) we obtain that in order for X, Y to form GCLP, the following two conditions are necessary and sufficient:

$$2\phi'^2(x) = \alpha_2\phi^2(x) + \alpha_6\phi(x) + \alpha_8 \tag{6.7}$$

and

$$U(x) = -\frac{\alpha_4\phi^2(x) + \alpha_7\phi(x) + \alpha_9}{\alpha_2\phi^2(x) + \alpha_6\phi(x) + \alpha_8}. \tag{6.8}$$

Condition (6.7) is a simple differential equation for the function $\phi(x)$. Condition (6.8) then gives possible potentials $U(x)$ admitted by the requirement that X, Y form GLP. To within affine transformations of the variable $x \to \beta_1 x + \beta_2$ (with *real* parameters β_1, β_2) there are essentially six types of the admitted potentials:

(i) hyperbolic Pöschl–Teller potential $u(x) = C_1 \tanh^2(x) + C_2 \tanh^{-2}(x) + C_3$;
(ii) modified hyperbolic Pöschl–Teller potential $u(x) = \left(C_1 \sinh^2 + C_2 \sinh(x) + C_3\right)/\cosh^2(x)$;
(iii) trigonometric Pöschl–Teller potential $u(x) = C_1 \tan^2(x) + C_2 \tan^{-2}(x) + C_3$;
(iv) Morse potential $u(x) = C_1 \,\mathrm{e}^{-2x} + C_2 \,\mathrm{e}^{-x} + C_3$
(v) singular oscillator $u(x) = C_1 x^2 + C_2 x^{-2} + C_3$
(vi) shifted oscillator $u(x) = C_1 x^2 + C_2 x + C_3$.

In all cases parameters C_1, C_2, C_3 can take any real values. The potentials of cases (i)–(iii) correspond to $\alpha_2 \neq 0$ and $\alpha_6^2 - 4\alpha_2\alpha_8 \neq 0$. The Morse potential (case (iv)) corresponds to $\alpha_2 \neq 0$ and $\alpha_6^2 - 4\alpha_2\alpha_8 = 0$. The singular oscillator potential (case (v)) corresponds to $\alpha_2 = 0, \alpha_6 \neq 0$. The shifted oscillator potential corresponds to $\alpha_2 = \alpha_6 = 0, \alpha_8 \neq 0$.

Note that AW-algebra corresponding to the cases (i)–(vi) is reduced to the so-called Jacobi algebra. For further details concerning this algebra and corresponding potentials (in both classical and quantum cases) see, e.g., [7].

Finally, consider a simple example of AW algebra connected with the classical $sl_q(2)$ algebra (in [9] we considered an operator version of this example). Recall that this algebra consists of three generators A_0, A_+, A_- and is defined by the relations

$$\{A_0, A_\pm\} = \pm A_\pm \qquad \{A_-, A_+\} = 2g\sinh(2\omega A_0) \tag{6.9}$$

where g and ω are arbitrary parameters. In the limit $\omega \to 0, g = 1/(2\omega)$ we arrive at the classical $sl(2)$ algebra

$$\{A_0, A_\pm\} = \pm A_\pm \qquad \{A_-, A_+\} = 2A_0. \tag{6.10}$$

The algebra $sl_q(2)$ has the Casimir element

$$q = A_+A_- - g\cosh(2\omega A_0)/\omega. \tag{6.11}$$

Assume that we have a representation of the classical $sl_q(2)$ algebra with fixed value q of the Casimir element q. Choose the variables

$$\begin{aligned} X &= (a_1A_+ + a_2A_- + a_3\exp(\omega A_0))\ \exp(\omega A_0) \\ Y &= (b_1A_+ + b_2A_- + b_3\exp(-\omega A_0))\ \exp(-\omega A_0). \end{aligned} \tag{6.12}$$

Using relations (6.9) we have

$$\begin{aligned} Z = \{X, Y\} = 2\omega &\left(a_1b_1A_+^2 - a_2b_2A_-^2 + \left(a_1b_3\,\mathrm{e}^{\omega A_0} + a_3b_1\,\mathrm{e}^{-\omega A_0}\right)A_+ \right. \\ &\left. - \left(a_2b_3\,\mathrm{e}^{\omega A_0} + a_3b_2\,\mathrm{e}^{-\omega A_0}\right)A_-\right) + 2g(a_2b_1 - a_1b_2)\sinh(2\omega A_0). \end{aligned} \tag{6.13}$$

Evaluating Z^2 and taking into account relation (6.11) we can verify that relation (2.18) is valid where

$$\begin{aligned}
&\alpha_1 = 4\omega^2 \qquad \alpha_2 = \alpha_3 = 0 \qquad \alpha_4 = -8b_1b_2g\omega \qquad \alpha_5 = -8a_1a_2g\omega \\
&\alpha_6 = -8\omega^2(q(a_1b_2 + a_2b_1) + a_3b_3) \\
&\alpha_7 = -8\omega(2\omega q b_1b_2a_3 + gb_3(a_1b_2 + a_2b_1)) \\
&\alpha_8 = -8\omega(2\omega q a_1a_2b_3 + ga_3(a_1b_2 + a_2b_1)) \\
&\alpha_9 = 4(-g^2 + q^2\omega^2)(a_1b_2 - b_1a_2)^2 + 4\omega^2a_3^2b_3^2 \\
&\qquad - 8g\omega\left(b_1b_2a_3^2 + a_1a_2b_3^2\right) - 8q\omega^2a_3b_3(a_1b_2 + a_2b_1).
\end{aligned} \tag{6.14}$$

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